

# *Non-Abelian Tensor Gauge Fields and Extended Current Algebra*

Generalization of Yang-Mills Theory

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## Abstract

We suggest an infinite-dimensional extension of the gauge transformations which includes non-Abelian tensor gauge fields. Extended gauge transformations of non-Abelian tensor gauge fields form a new large group which has natural geometrical interpretation in terms of extended current algebra associated with compact Lie group. We shall demonstrate that one can construct two infinite series of gauge invariant quadratic forms, so that a linear combination of them comprises the general Lagrangian. The general Lagrangian exhibits enhanced local gauge invariants with double number of gauge parameters and allows to eliminate all negative norm states of the nonsymmetric second-rank tensor gauge field. Therefore it describes two polarizations of helicity-two and helicity-zero massless charged tensor gauge bosons.

# 1 Introduction

The non-Abelian local gauge invariance, which was formulated by Yang and Mills in [1], requires that all interactions must be invariant under independent rotations of internal charges at all space-time points. The gauge principle allows very little arbitrariness: the interaction of matter fields which carry non-commuting internal charges and the nonlinear self-interaction of gauge bosons are essentially fixed by the requirement of local gauge invariance, very similar to the self-interaction of gravitons in general relativity.

It is therefore appealing to extend the gauge principle, which was elevated by Yang and Mills to a powerful constructive principle, so that it will define the interaction of matter fields which carry not only non-commutative internal charges, but also arbitrary large spins. It seems that this will naturally lead to a theory in which fundamental forces will be mediated by integer-spin gauge quanta and that the Yang-Mills vector gauge boson will become a member of a bigger family of tensor gauge bosons.

In the previous papers [2, 3] we extended the gauge principle so that it enlarges the original algebra of the Abelian local gauge transformations found in [4, 5, 6] to a non-Abelian case. The extended non-Abelian gauge transformations of the tensor gauge fields form *a new large group which has a natural geometrical interpretation in terms of extended current algebra associated with the Lorentz group*. On this large group one can define field strength tensors, which are transforming homogeneously with respect to the extended gauge transformations. The invariant Lagrangian is quadratic in the field strength tensors and describes interaction of tensor gauge fields of arbitrary large integer spin  $1, 2, \dots$

We shall present a second invariant Lagrangian which can be constructed in terms of the above field strength tensors. The total Lagrangian is a sum of the two Lagrangians and exhibits enhanced local gauge invariance with double number of gauge parameters. This allows to eliminate all negative norm states of the nonsymmetric second rank tensor gauge field  $A_{\mu\lambda}$ , which describes therefore two polarizations of helicity-two massless charged tensor gauge boson and of the helicity-zero "axion".

The early investigation of higher-spin representations of the Poincaré algebra and of the corresponding field equations is due to Majorana, Dirac and Wigner [7, 8, 10]. The theory of massive particles of higher spin was further developed by Fierz and Pauli [9] and Rarita and Schwinger [11]. The Lagrangian and S-matrix formulations of *free field theory* of massive and massless fields with higher spin have been completely constructed in [12, 13, 14, 15, 16, 17, 18, 19, 20, 21]. The problem of *introducing interaction* appears to be much more complex [22, 23, 24, 25, 26, 27, 33, 34] and met enormous difficulties for spin fields higher than two [28, 29, 30, 31, 32]. The first positive result in this direction was the light-front construction of the cubic interaction term for the massless field of helicity  $\pm\lambda$  in [35, 36].

In our approach the gauge fields are defined as rank- $(s+1)$  tensors [2, 3]

$$A_{\mu\lambda_1\dots\lambda_s}^a(x), \quad s = 0, 1, 2, \dots$$

and are totally symmetric with respect to the indices  $\lambda_1\dots\lambda_s$ . *A priori* the tensor gauge fields have no symmetries with respect to the first index  $\mu$ . The index  $a$  numerates the generators  $L^a$  of the Lie algebra  $\mathfrak{g}$  of a *compact*<sup>1</sup> Lie group  $G$ . One can think of these

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<sup>1</sup>The algebra  $\mathfrak{g}$  possesses an orthogonal basis in which the structure constant  $f^{abc}$  are totally antisymmetric.

tensor gauge fields as appearing in the expansion of the extended gauge field  $\mathcal{A}_\mu(x, e)$  over the coordinates  $e^\lambda$  of the tangent space [2, 3, 5]:

$$\mathcal{A}_\mu(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} A_{\mu\lambda_1\ldots\lambda_s}^a(x) L^a e^{\lambda_1} \ldots e^{\lambda_s}. \quad (1)$$

In this sense the gauge field  $A_{\mu\lambda_1\ldots\lambda_s}^a$  carries extra indices  $\lambda_1, \ldots, \lambda_s$ , which together with index  $a$  are labeling the generators  $L_{\lambda_1\ldots\lambda_s}^a$  of *extended current algebra*  $\mathcal{G}$  associated with the Lorentz group. The corresponding algebra has infinite many generators  $L_{\lambda_1\ldots\lambda_s}^a = L^a e_{\lambda_1} \ldots e_{\lambda_s}$  and is given by the commutator

$$[L_{\lambda_1\ldots\lambda_s}^a, L_{\rho_1\ldots\rho_k}^b] = i f^{abc} L_{\lambda_1\ldots\lambda_s\rho_1\ldots\rho_k}^c. \quad (2)$$

The extended non-Abelian gauge transformations of the tensor gauge fields are defined by the following equations [2, 3]:

$$\begin{aligned} \delta A_\mu^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi^b, \\ (I) \quad \delta A_{\mu\nu}^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi_\nu^b + g f^{acb} A_{\mu\nu}^c \xi^b, \\ \delta A_{\mu\nu\lambda}^a &= (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi_{\nu\lambda}^b + g f^{acb} (A_{\mu\nu}^c \xi_\lambda^b + A_{\mu\lambda}^c \xi_\nu^b + A_{\nu\lambda}^c \xi_\mu^b), \\ &\dots\dots\dots \end{aligned} \quad (3)$$

or in a general form by the formula

$$\delta A_{\mu\lambda_1\ldots\lambda_s}^a = (\delta^{ab} \partial_\mu + g f^{acb} A_\mu^c) \xi_{\lambda_1\lambda_2\ldots\lambda_s}^b + g f^{acb} \sum_{i=1}^s \sum_{P's} A_{\mu\lambda_1\ldots\lambda_i}^c \xi_{\lambda_{i+1}\ldots\lambda_s}^b, \quad (4)$$

where the infinitesimal gauge parameters  $\xi_{\lambda_1\ldots\lambda_s}^b$  are totally symmetric rank- $s$  tensors. The summation  $\sum_{P's}$  is over all permutations of two sets of indices  $\lambda_1\ldots\lambda_i$  and  $\lambda_{i+1}\ldots\lambda_s$  which correspond to nonequal terms. It is obvious that this transformation preserves the symmetry properties of the tensor gauge field  $A_{\mu\lambda_1\ldots\lambda_s}^a$ . Indeed, the first term in the r.h.s. is a covariant derivative of the totally symmetric rank- $s$  tensor  $\nabla_\mu^{ab} \xi_{\lambda_1\lambda_2\ldots\lambda_s}^b$  and every term  $\sum_{P's} A_{\mu\lambda_1\ldots\lambda_i}^c \xi_{\lambda_{i+1}\ldots\lambda_s}^b$  in the second sum is totally symmetric with respect to the indices  $\lambda_1\lambda_2\ldots\lambda_s$  by construction. The matrix form of the transformation is

$$\delta_\xi A_{\mu\lambda_1\ldots\lambda_s} = \partial_\mu \xi_{\lambda_1\lambda_2\ldots\lambda_s} - ig [A_\mu, \xi_{\lambda_1\lambda_2\ldots\lambda_s}] - ig \sum_{i=1}^s \sum_{P's} [A_{\mu\lambda_1\ldots\lambda_i}, \xi_{\lambda_{i+1}\ldots\lambda_s}], \quad (5)$$

where the tensor gauge fields are  $A_{\mu\lambda_1\ldots\lambda_s}^{ab} = (L_c)^{ab} A_{\mu\lambda_1\ldots\lambda_s}^c = i f^{acb} A_{\mu\lambda_1\ldots\lambda_s}^c$ , and  $L^a$  are the generators of the compact Lie group  $G$  in the adjoint representation.

These extended gauge transformations generate a closed algebraic structure. To see that, one should compute the commutator of two extended gauge transformations  $\delta_\eta$  and  $\delta_\xi$  of parameters  $\eta$  and  $\xi$ . The commutator of two transformations can be expressed in the form [2, 3]

$$[\delta_\eta, \delta_\xi] A_{\mu\lambda_1\lambda_2\ldots\lambda_s} = -ig \delta_\zeta A_{\mu\lambda_1\lambda_2\ldots\lambda_s} \quad (6)$$

and is again an extended gauge transformation with the gauge parameters  $\{\zeta\}$  which are given by the matrix commutators

$$\begin{aligned} \zeta &= [\eta, \xi] \\ \zeta_{\lambda_1} &= [\eta, \xi_{\lambda_1}] + [\eta_{\lambda_1}, \xi] \\ \zeta_{\nu\lambda} &= [\eta, \xi_{\nu\lambda}] + [\eta_\nu, \xi_\lambda] + [\eta_\lambda, \xi_\nu] + [\eta_{\nu\lambda}, \xi], \\ &\dots\dots\dots \end{aligned} \quad (7)$$

The generalized field strengths are defined as [2, 3]

$$\begin{aligned}
G_{\mu\nu}^a &= \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c, \\
G_{\mu\nu,\lambda}^a &= \partial_\mu A_{\nu\lambda}^a - \partial_\nu A_{\mu\lambda}^a + gf^{abc} (A_\mu^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_\nu^c), \\
G_{\mu\nu,\lambda\rho}^a &= \partial_\mu A_{\nu\lambda\rho}^a - \partial_\nu A_{\mu\lambda\rho}^a + gf^{abc} (A_\mu^b A_{\nu\lambda\rho}^c + A_{\mu\lambda}^b A_{\nu\rho}^c + A_{\mu\rho}^b A_{\nu\lambda}^c + A_{\mu\lambda\rho}^b A_\nu^c), \\
&\dots \quad \cdot \quad \dots\dots\dots
\end{aligned} \tag{8}$$

and transform homogeneously with respect to the extended gauge transformations (3). The field strength tensors are antisymmetric in their first two indices and are totally symmetric with respect to the rest of the indices. By induction the entire construction can be generalized to include tensor fields of any rank  $s$ , and the corresponding field strength we shall define by the following expression:

$$G_{\mu\nu,\lambda_1\dots\lambda_s}^a = \partial_\mu A_{\nu\lambda_1\dots\lambda_s}^a - \partial_\nu A_{\mu\lambda_1\dots\lambda_s}^a + gf^{abc} \sum_{i=0}^s \sum_{P's} A_{\mu\lambda_1\dots\lambda_i}^b A_{\nu\lambda_{i+1}\dots\lambda_s}^c, \tag{9}$$

where the sum  $\sum_{P's}$  runs over all permutations of two sets of indices  $\lambda_1\dots\lambda_i$  and  $\lambda_{i+1}\dots\lambda_s$  which correspond to nonequal terms. All permutations of indices within two sets  $\lambda_1\dots\lambda_i$  and  $\lambda_{i+1}\dots\lambda_s$  correspond to equal terms, because gauge fields are totally symmetric with respect to  $\lambda_1\dots\lambda_i$  and  $\lambda_{i+1}\dots\lambda_s$ . Therefore there are

$$\frac{s!}{i!(s-i)!}$$

nonequal terms in the sum  $\sum_{P's}$ . Thus in the sum  $\sum_{P's}$  there is one term of the form  $A_\mu A_{\nu\lambda_1\lambda_2\dots\lambda_s}$ , there are  $s$  terms of the form  $A_{\mu\lambda_1} A_{\nu\lambda_2\dots\lambda_s}$  and  $s(s-1)/2$  terms of the form  $A_{\mu\lambda_1\lambda_2} A_{\nu\lambda_3\dots\lambda_s}$  and so on. In the above definition of the extended gauge field strength  $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$ , together with the classical Yang-Mills gauge boson  $A_\mu^a$ , there participate a set of higher-rank gauge fields  $A_{\mu\lambda_1}^a, A_{\mu\lambda_1\lambda_2}^a, \dots, A_{\mu\lambda_1\dots\lambda_s}^a$  up to the rank  $s+1$ . By construction the field strength (9) is antisymmetric with respect to its first two indices  $G_{\mu\nu,\lambda_1\dots\lambda_s}^a = -G_{\nu\mu,\lambda_1\dots\lambda_s}^a$  and is totally symmetric with respect to the rest of the indices  $G_{\mu\nu,\dots\lambda_i\dots\lambda_j\dots\lambda_s}^a = G_{\mu\nu,\dots\lambda_j\dots\lambda_i\dots\lambda_s}^a$ , where  $1 \leq i < j \leq s$ .

The inhomogeneous extended gauge transformation (4) induces the homogeneous gauge transformation of the corresponding field strength (9) of the form [2, 3]

$$\begin{aligned}
\delta G_{\mu\nu}^a &= gf^{abc} G_{\mu\nu}^b \xi^c \\
\delta G_{\mu\nu,\lambda}^a &= gf^{abc} (G_{\mu\nu,\lambda}^b \xi^c + G_{\mu\nu}^b \xi_\lambda^c), \\
\delta G_{\mu\nu,\lambda\rho}^a &= gf^{abc} (G_{\mu\nu,\lambda\rho}^b \xi^c + G_{\mu\nu,\lambda}^b \xi_\rho^c + G_{\mu\nu,\rho}^b \xi_\lambda^c + G_{\mu\nu}^b \xi_{\lambda\rho}^c) \\
&\dots \quad \cdot \quad \dots\dots\dots,
\end{aligned} \tag{10}$$

or in general

$$\delta G_{\mu\nu,\lambda_1\dots\lambda_s}^a = gf^{abc} \sum_{i=0}^s \sum_{P's} G_{\mu\nu,\lambda_1\dots\lambda_i}^b \xi_{\lambda_{i+1}\dots\lambda_s}^c. \tag{11}$$

The symmetry properties of the field strength  $G_{\mu\nu,\lambda_1\dots\lambda_s}^a$  remain invariant in the course of this transformation.

The gauge invariant Lagrangian now can be formulated in the form [2, 3]

$$\begin{aligned}\mathcal{L}_{s+1} &= -\frac{1}{4} G_{\mu\nu,\lambda_1\ldots\lambda_s}^a G_{\mu\nu,\lambda_1\ldots\lambda_s}^a + \dots\dots \\ &= -\frac{1}{4} \sum_{i=0}^{2s} a_i^s G_{\mu\nu,\lambda_1\ldots\lambda_i}^a G_{\mu\nu,\lambda_{i+1}\ldots\lambda_{2s}}^a \left( \sum_{p's} \eta^{\lambda_{i_1}\lambda_{i_2}} \dots \eta^{\lambda_{i_{2s-1}}\lambda_{i_{2s}}} \right),\end{aligned}\quad (12)$$

where the sum  $\sum_p$  runs over all nonequal permutations of  $i$ 's, in total  $(2s-1)!!$  terms. For the low values of  $s = 0, 1, 2, \dots$  the numerical coefficients and eta functions are

$$a_i^s = \frac{s!}{i!(2s-i)!}$$

$a_0^0 = 1$ ;  $a_1^1 = 1, a_0^1 = a_2^1 = 1/2$ ;  $a_2^2 = 1/2, a_1^2 = a_3^2 = 1/3, a_0^2 = a_4^2 = 1/12$  and  $\eta^{\lambda_1\lambda_2}, \eta^{\lambda_1\lambda_2}\eta^{\lambda_3\lambda_4} + \eta^{\lambda_1\lambda_3}\eta^{\lambda_2\lambda_4} + \eta^{\lambda_1\lambda_4}\eta^{\lambda_2\lambda_3}, \dots$  and so on. In order to describe fixed rank- $(s+1)$  gauge field one should have at disposal all gauge fields up to the rank  $2s+1$ . In order to make all tensor gauge fields dynamical one should add the corresponding kinetic terms. Thus the invariant Lagrangian describing dynamical tensor gauge bosons of all ranks has the form

$$\mathcal{L} = \sum_{s=1}^{\infty} g_s \mathcal{L}_s. \quad (13)$$

The first three terms of the invariant Lagrangian have the following form [2, 3]:

$$\begin{aligned}\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \dots = & -\frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a - \frac{1}{4} G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda}^a - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda}^a - \\ & -\frac{1}{4} G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - \frac{1}{8} G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a - \frac{1}{2} G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda\rho\rho}^a - \frac{1}{8} G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda\rho\rho}^a + \dots,\end{aligned}\quad (14)$$

where the first term is the Yang-Mills Lagrangian and the second and the third ones describe the tensor gauge fields  $A_{\mu\nu}^a, A_{\mu\nu\lambda}^a$  and so on. It is important that: i) *the Lagrangian does not contain higher derivatives of tensor gauge fields* ii) *all interactions take place through the three- and four-particle exchanges with dimensionless coupling constant* iii) *the complete Lagrangian contains all higher-rank tensor gauge fields and should not be truncated.*

## 2 Geometrical Interpretation

Let us consider a possible geometrical interpretation of the above construction. Introducing the coordinates  $e^\mu$  on the tangent space we shall consider non-Abelian gauge transformations  $U(\xi)$  with the extended gauge parameter  $\xi(x, e)$  which depends on the space-time coordinates  $x^\mu$  and the tangent coordinates  $e^\lambda$ . We can expand the gauge parameter  $\xi(x, e)$  in series using generators  $L_{\lambda_1\ldots\lambda_s}^a = L^a e_{\lambda_1}\ldots e_{\lambda_s}$  [2, 3, 5]

$$\xi(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} \xi_{\lambda_1\ldots\lambda_s}^a(x) L^a e_{\lambda_1}\ldots e_{\lambda_s} \quad (15)$$

and define the gauge transformation of the extended gauge field  $\mathcal{A}_\mu(x, e)$  as in (3)

$$\mathcal{A}'_\mu(x, e) = U(\xi) \mathcal{A}_\mu(x, e) U^{-1}(\xi) - \frac{i}{g} \partial_\mu U(\xi) U^{-1}(\xi), \quad (16)$$

where the unitary transformation matrix is given by the expression

$$U(\xi) = \exp\{ig\xi(x, e)\}.$$

This allows to construct the extended field strength tensor of the form (8)

$$\mathcal{G}_{\mu\nu}(x, e) = \partial_\mu \mathcal{A}_\nu(x, e) - \partial_\nu \mathcal{A}_\mu(x, e) - ig[\mathcal{A}_\mu(x, e) \mathcal{A}_\nu(x, e)] \quad (17)$$

using the commutator of the covariant derivatives

$$\nabla_\mu^{ab} = (\partial_\mu - ig\mathcal{A}_\mu(x, e))^{ab}$$

of a standard form  $[\nabla_\mu, \nabla_\nu]^{ab} = gf^{acb}\mathcal{G}_{\mu\nu}^c$ , so that

$$\mathcal{G}'_{\mu\nu}(x, e) = U(\xi)\mathcal{G}_{\mu\nu}(x, e)U^{-1}(\xi). \quad (18)$$

The invariant Lagrangian density is given by the expression

$$\mathcal{L}(x, e) = -\frac{1}{4}\mathcal{G}_{\mu\nu}^a(x, e)\mathcal{G}_{\mu\nu}^a(x, e), \quad (19)$$

as one can get convinced computing its variation with respect to the extended gauge transformation (3),(16) and (10),(18)

$$\delta\mathcal{L}(x, e) = -\frac{1}{2}\mathcal{G}_{\mu\nu}^a(x, e) gf^{acb} \mathcal{G}_{\mu\nu}^c(x, e) \xi^b(x, e) = 0.$$

The Lagrangian density (19) allows to extract *gauge invariant, totally symmetric, tensor densities*  $\mathcal{L}_{\lambda_1 \dots \lambda_s}(x)$  using expansion with respect to the vector variable  $e^\lambda$

$$\mathcal{L}(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} \mathcal{L}_{\lambda_1 \dots \lambda_s}(x) e_{\lambda_1} \dots e_{\lambda_s}. \quad (20)$$

In particular the expansion term which is quadratic in powers of  $e$  is

$$(\mathcal{L}_2)_{\lambda\rho} = -\frac{1}{4}G_{\mu\nu,\lambda}^a G_{\mu\nu,\rho}^a - \frac{1}{4}G_{\mu\nu}^a G_{\mu\nu,\lambda\rho}^a \quad (21)$$

and defines a unique gauge invariant Lagrangian which can be constructed from the above tensor (see the next section for its explicit variation (26)), that is the Lagrangian  $\mathcal{L}_2$

$$\mathcal{L}_2 = -\frac{1}{4}G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda}^a - \frac{1}{4}G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda}^a$$

and so on.

The whole construction can be viewed as an extended vector bundle  $X$  on which the gauge field  $\mathcal{A}_\mu^a(x, e)$  is a connection. In this sense the gauge field  $A_{\mu\lambda_1 \dots \lambda_s}^a$  carries extra indices  $\lambda_1, \dots, \lambda_s$ , which together with index  $a$  are labeling the generators  $L_{\lambda_1 \dots \lambda_s}^a$  of *extended current algebra*  $\mathcal{G}$  associated with the Lorentz group. The corresponding algebra has infinite many generators  $L_{\lambda_1 \dots \lambda_s}^a = L^a e_{\lambda_1} \dots e_{\lambda_s}$  and is given by the commutator

$$[L_{\lambda_1 \dots \lambda_s}^a, L_{\rho_1 \dots \rho_k}^b] = if^{abc} L_{\lambda_1 \dots \lambda_s \rho_1 \dots \rho_k}^c. \quad (22)$$

Thus we have vector bundle whose structure group is an extended gauge group  $\mathcal{G}$  with group elements  $U(\xi) = \exp(i\xi(e))$ , where  $\xi(e) = \sum_s \xi_{\lambda_1 \dots \lambda_s}^a L_{\lambda_1 \dots \lambda_s}^a e_{\lambda_1} \dots e_{\lambda_s}$  and the composition law (7). In contrast, in Kac-Moody current algebra the generators depend on the complex variable  $L_n^a = L^a z^n$  (see also [58])

$$[L_n^a, L_m^b] = if^{abc} L_{n+m}^c.$$

In the next section we shall see that, there exist a second invariant Lagrangian  $\mathcal{L}'$  which can be constructed in terms of extended field strength tensors (8) and the total Lagrangian is a linear sum of the two Lagrangians  $c\mathcal{L} + c'\mathcal{L}'$ .

### 3 Enhanced Local Gauge Symmetry

Indeed the Lagrangian (12), (13) and (14) is not the most general Lagrangian which can be constructed in terms of the above field strength tensors (8) and (9). As we shall see there exists a second invariant Lagrangian  $\mathcal{L}'$  (23), (24) and (25) which can be constructed in terms of extended field strength tensors (8) and the total Lagrangian is a linear sum of the two Lagrangians  $c \mathcal{L} + c' \mathcal{L}'$ . In particular for the second-rank tensor gauge field  $A_{\mu\lambda}^a$  the total Lagrangian is a sum of two Lagrangians  $\mathcal{L}_2 + \mathcal{L}'_2$  and, with specially chosen coefficients  $\{c, c'\}$ , it exhibits an enhanced gauge invariance (31), (62) with double number of gauge parameters, which allows to eliminate negative norm polarizations of the nonsymmetric second-rank tensor gauge field  $A_{\mu\lambda}$ . The geometrical interpretation of the enhanced gauge symmetry with double number of gauge parameters is not yet known.

Let us consider the gauge invariant tensor density of the form

$$\mathcal{L}'_{\rho_1\rho_2}(x, e) = \frac{1}{4} \mathcal{G}_{\mu\rho_1}^a(x, e) \mathcal{G}_{\mu\rho_2}^a(x, e). \quad (23)$$

It is gauge invariant because its variation is also equal to zero

$$\delta \mathcal{L}'_{\rho_1\rho_2}(x, e) = \frac{1}{4} g f^{acb} \mathcal{G}_{\mu\rho_1}^c(x, e) \xi^b(x, e) \mathcal{G}_{\mu\rho_2}^a(x, e) + \frac{1}{4} \mathcal{G}_{\mu\rho_1}^a(x, e) g f^{acb} \mathcal{G}_{\mu\rho_2}^c(x, e) \xi^b(x, e) = 0.$$

The Lagrangian density (23) generate the second series of *gauge invariant tensor densities*  $(\mathcal{L}'_{\rho_1\rho_2})_{\lambda_1 \dots \lambda_s}(x)$  when we expand it in powers of the vector variable  $e$

$$\mathcal{L}'_{\rho_1\rho_2}(x, e) = \sum_{s=0}^{\infty} \frac{1}{s!} (\mathcal{L}'_{\rho_1\rho_2})_{\lambda_1 \dots \lambda_s}(x) e_{\lambda_1} \dots e_{\lambda_s}. \quad (24)$$

Using contraction of these tensor densities the gauge invariant Lagrangians can be formulated in the form

$$\begin{aligned} \mathcal{L}'_{s+1} &= \frac{1}{4} G_{\mu\lambda_1, \lambda_2 \dots \lambda_{s+1}}^a G_{\mu\lambda_2, \lambda_1 \dots \lambda_{s+1}}^a + \frac{1}{4} G_{\mu\nu, \nu\lambda_3 \dots \lambda_{s+1}}^a G_{\mu\rho, \rho\lambda_3 \dots \lambda_{s+1}}^a + \dots \\ &= \frac{1}{4} \sum_{i=1}^{2s+1} \frac{a_{i-1}^s}{s} G_{\mu\lambda_1, \lambda_2 \dots \lambda_i}^a G_{\mu\lambda_{i+1}, \lambda_{i+2} \dots \lambda_{2s+2}}^a \left( \sum_{p's} \eta^{\lambda_{i1} \lambda_{i2}} \dots \eta^{\lambda_{i2s+1} \lambda_{i2s+2}} \right), \end{aligned} \quad (25)$$

where the sum  $\sum_{p's}$  runs over all nonequal permutations of  $i's$ , with exclusion of the terms which contain  $\eta^{\lambda_1, \lambda_{i+1}}$ .

It is important to consider these Lagrangians in details. The invariance of the Lagrangian

$$\mathcal{L}_2 = -\frac{1}{4} G_{\mu\nu, \lambda}^a G_{\mu\nu, \lambda}^a - \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu, \lambda\lambda}^a$$

in (12), (13) and (14) was demonstrated in [3] by calculating its variation with respect to the gauge transformation (3) and (10), (11). Indeed, its variation is equal to zero

$$\begin{aligned} \delta \mathcal{L}_2 &= -\frac{1}{4} G_{\mu\nu, \lambda}^a g f^{abc} (G_{\mu\nu, \lambda}^b \xi^c + G_{\mu\nu}^b \xi_{\lambda}^c) - \frac{1}{4} g f^{abc} (G_{\mu\nu, \lambda}^a \xi^c + G_{\mu\nu}^b \xi_{\lambda}^c) G_{\mu\nu, \lambda}^b \\ &\quad - \frac{1}{4} g f^{abc} G_{\mu\nu}^b \xi^c G_{\mu\nu, \lambda\lambda}^a \\ &\quad - \frac{1}{4} G_{\mu\nu}^a g f^{abc} (G_{\mu\nu, \lambda\lambda}^b \xi^c + G_{\mu\nu, \lambda}^b \xi_{\lambda}^c + G_{\mu\nu, \lambda}^b \xi_{\lambda}^c + G_{\mu\nu}^b \xi_{\lambda\lambda}^c) = 0. \end{aligned} \quad (26)$$

As we have seen the Lagrangian  $\mathcal{L}$  (13) is not a unique one and that there exist a second series of invariants  $\mathcal{L}'$  (25). Let us construct few of them directly, without reference to any expansion. Indeed, there are three Lorentz invariants in our disposal

$$G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu}^a, \quad G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda}^a, \quad G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^a.$$

Calculating the variation of each of these terms with respect to the gauge transformation (3) and (10) one can get convinced that a particular linear combination

$$\mathcal{L}'_2 = \frac{1}{4} G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu}^a + \frac{1}{4} G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda}^a + \frac{1}{2} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^a \quad (27)$$

forms an invariant form which coincides with (25) for  $s=1$ . The variation of the Lagrangian  $\mathcal{L}'_2$  under the gauge transformation (10) is equal to zero:

$$\begin{aligned} \delta\mathcal{L}'_2 = & + \frac{1}{4} G_{\mu\nu,\lambda}^a g f^{abc} (G_{\mu\lambda,\nu}^b \xi^c + G_{\mu\lambda}^b \xi_\nu^c) + \frac{1}{4} g f^{abc} (G_{\mu\nu,\lambda}^b \xi^c + G_{\mu\nu}^b \xi_\lambda^c) G_{\mu\lambda,\nu}^a \\ & + \frac{1}{2} G_{\mu\nu,\nu}^a g f^{abc} (G_{\mu\lambda,\lambda}^b \xi^c + G_{\mu\lambda}^b \xi_\lambda^c) \\ & + \frac{1}{2} g f^{abc} G_{\mu\nu}^b \xi^c G_{\mu\lambda,\nu\lambda}^a \\ & + \frac{1}{2} G_{\mu\nu}^a g f^{abc} (G_{\mu\lambda,\nu\lambda}^b \xi^c + G_{\mu\lambda,\nu}^b \xi_\lambda^c + G_{\mu\lambda,\lambda}^b \xi_\nu^c + G_{\mu\lambda}^b \xi_{\nu\lambda}^c) = 0. \end{aligned}$$

As a result we have two invariant Lagrangians  $\mathcal{L}_2$  and  $\mathcal{L}'_2$  and the general Lagrangian is a linear combination of these two Lagrangians  $\mathcal{L}_2 + c\mathcal{L}'_2$ , where  $c$  is an arbitrary constant.

*Our aim now is to demonstrate that if  $c = 1$  then we shall have enhanced local gauge invariance (31), (62) of the Lagrangian  $\mathcal{L}_2 + \mathcal{L}'_2$  with double number of gauge parameters. This allows to eliminate all negative norm states of the nonsymmetric second-rank tensor gauge field  $A_{\mu\lambda}^a$ , which describes therefore two polarizations of helicity-two and helicity-zero massless charged tensor gauge bosons.*

Indeed, let us consider the situation at the linearized level when the gauge coupling constant  $g$  is equal to zero. The free part of the  $\mathcal{L}_2$  Lagrangian is

$$\mathcal{L}_2^{free} = \frac{1}{2} A_{\alpha\dot{\alpha}}^a (\eta_{\alpha\gamma} \eta_{\dot{\alpha}\dot{\gamma}} \partial^2 - \eta_{\dot{\alpha}\dot{\gamma}} \partial_\alpha \partial_\gamma) A_{\gamma\dot{\gamma}}^a = \frac{1}{2} A_{\alpha\dot{\alpha}}^a H_{\alpha\dot{\alpha}\gamma\dot{\gamma}} A_{\gamma\dot{\gamma}}^a,$$

where the quadratic form in the momentum representation has the form

$$H_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k) = -(k^2 \eta_{\alpha\gamma} - k_\alpha k_\gamma) \eta_{\dot{\alpha}\dot{\gamma}} = -H_{\alpha\gamma}(k) \eta_{\dot{\alpha}\dot{\gamma}},$$

is obviously invariant with respect to the gauge transformation  $\delta A_{\mu\lambda}^a = \partial_\mu \xi_\lambda^a$ , but it is not invariant with respect to the alternative gauge transformations  $\delta A_{\mu\lambda}^a = \partial_\lambda \eta_\mu^a$ . This can be seen, for example, from the following relations in the momentum representation:

$$k_\alpha H_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k) = 0, \quad k_{\dot{\alpha}} H_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k) = -(k^2 \eta_{\alpha\gamma} - k_\alpha k_\gamma) k_{\dot{\gamma}} \neq 0. \quad (28)$$

Let us consider now the free part of the second Lagrangian

$$\begin{aligned} \mathcal{L}_2'^{free} = & \frac{1}{4} A_{\alpha\dot{\alpha}}^a (-\eta_{\alpha\dot{\gamma}} \eta_{\dot{\alpha}\gamma} \partial^2 - \eta_{\alpha\dot{\alpha}} \eta_{\gamma\dot{\gamma}} \partial^2 + \eta_{\alpha\dot{\gamma}} \partial_{\dot{\alpha}} \partial_\gamma + \eta_{\dot{\alpha}\gamma} \partial_\alpha \partial_{\dot{\gamma}} + \eta_{\alpha\dot{\alpha}} \partial_\gamma \partial_{\dot{\gamma}} + \\ & + \eta_{\gamma\dot{\gamma}} \partial_\alpha \partial_{\dot{\alpha}} - 2\eta_{\alpha\gamma} \partial_{\dot{\alpha}} \partial_{\dot{\gamma}}) A_{\gamma\dot{\gamma}}^a = \frac{1}{2} A_{\alpha\dot{\alpha}}^a H_{\alpha\dot{\alpha}\gamma\dot{\gamma}}' A_{\gamma\dot{\gamma}}^a, \end{aligned} \quad (29)$$



where

$$H'_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k) = \frac{1}{2}(\eta_{\alpha\dot{\gamma}}\eta_{\dot{\alpha}\gamma} + \eta_{\alpha\dot{\alpha}}\eta_{\gamma\dot{\gamma}})k^2 - \frac{1}{2}(\eta_{\alpha\dot{\gamma}}k_{\dot{\alpha}}k_{\gamma} + \eta_{\dot{\alpha}\gamma}k_{\alpha}k_{\dot{\gamma}} + \eta_{\alpha\dot{\alpha}}k_{\gamma}k_{\dot{\gamma}} + \eta_{\gamma\dot{\gamma}}k_{\alpha}k_{\dot{\alpha}} - 2\eta_{\alpha\gamma}k_{\dot{\alpha}}k_{\dot{\gamma}}).$$

It is again invariant with respect to the gauge transformation  $\delta A_{\mu\lambda}^a = \partial_{\mu}\xi_{\lambda}^a$ , but it is not invariant with respect to the gauge transformations  $\delta A_{\mu\lambda}^a = \partial_{\lambda}\eta_{\mu}^a$  as one can see from analogous relations

$$k_{\alpha}H'_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k) = 0, \quad k_{\dot{\alpha}}H'_{\alpha\dot{\alpha}\gamma\dot{\gamma}}(k) = (k^2\eta_{\alpha\gamma} - k_{\alpha}k_{\gamma})k_{\dot{\gamma}} \neq 0. \quad (30)$$

As it is obvious from (28) and (30), the total Lagrangian  $\mathcal{L}_2^{free} + \mathcal{L}_2'^{free}$  now poses new enhanced invariance with respect to the larger, eight parameter, gauge transformations

$$\delta A_{\mu\lambda}^a = \partial_{\mu}\xi_{\lambda}^a + \partial_{\lambda}\eta_{\mu}^a + \dots, \quad (31)$$

where  $\xi_{\lambda}^a$  and  $\eta_{\mu}^a$  are eight arbitrary functions, because

$$k_{\alpha}(H_{\alpha\dot{\alpha}\gamma\dot{\gamma}} + H'_{\alpha\dot{\alpha}\gamma\dot{\gamma}}) = 0, \quad k_{\dot{\alpha}}(H_{\alpha\dot{\alpha}\gamma\dot{\gamma}} + H'_{\alpha\dot{\alpha}\gamma\dot{\gamma}}) = 0. \quad (32)$$

Thus our free part of the Lagrangian is

$$\begin{aligned} \mathcal{L}_2^{tot \ free} = & - \frac{1}{2}\partial_{\mu}A_{\nu\lambda}^a\partial_{\mu}A_{\nu\lambda}^a + \frac{1}{2}\partial_{\mu}A_{\nu\lambda}^a\partial_{\nu}A_{\mu\lambda}^a + \\ & + \frac{1}{4}\partial_{\mu}A_{\nu\lambda}^a\partial_{\mu}A_{\lambda\nu}^a - \frac{1}{4}\partial_{\mu}A_{\nu\lambda}^a\partial_{\lambda}A_{\mu\nu}^a - \frac{1}{4}\partial_{\nu}A_{\mu\lambda}^a\partial_{\mu}A_{\lambda\nu}^a + \frac{1}{4}\partial_{\nu}A_{\mu\lambda}^a\partial_{\lambda}A_{\mu\nu}^a \\ & + \frac{1}{4}\partial_{\mu}A_{\nu\nu}^a\partial_{\mu}A_{\lambda\lambda}^a - \frac{1}{2}\partial_{\mu}A_{\nu\nu}^a\partial_{\lambda}A_{\mu\lambda}^a + \frac{1}{4}\partial_{\nu}A_{\mu\nu}^a\partial_{\lambda}A_{\mu\lambda}^a \end{aligned} \quad (33)$$

or, in equivalent form, it is

$$\begin{aligned} \mathcal{L}_2^{tot \ free} = & \frac{1}{2}A_{\alpha\dot{\alpha}}^a\{(\eta_{\alpha\gamma}\eta_{\dot{\alpha}\dot{\gamma}} - \frac{1}{2}\eta_{\alpha\dot{\gamma}}\eta_{\dot{\alpha}\gamma} - \frac{1}{2}\eta_{\alpha\dot{\alpha}}\eta_{\gamma\dot{\gamma}})\partial^2 - \eta_{\dot{\alpha}\dot{\gamma}}\partial_{\alpha}\partial_{\gamma} - \eta_{\alpha\gamma}\partial_{\dot{\alpha}}\partial_{\dot{\gamma}} + \\ & + \frac{1}{2}(\eta_{\alpha\dot{\gamma}}\partial_{\dot{\alpha}}\partial_{\gamma} + \eta_{\dot{\alpha}\gamma}\partial_{\alpha}\partial_{\dot{\gamma}} + \eta_{\alpha\dot{\alpha}}\partial_{\gamma}\partial_{\dot{\gamma}} + \eta_{\gamma\dot{\gamma}}\partial_{\alpha}\partial_{\dot{\alpha}})\}A_{\gamma\dot{\gamma}}^a \end{aligned} \quad (34)$$

and is invariant with respect to the larger gauge transformations  $\delta A_{\mu\lambda}^a = \partial_{\mu}\xi_{\lambda}^a + \partial_{\lambda}\eta_{\mu}^a$ , where  $\xi_{\lambda}^a$  and  $\eta_{\mu}^a$  are eight arbitrary functions. In the momentum representation the quadratic form is

$$\begin{aligned} H_{\alpha\dot{\alpha}\gamma\dot{\gamma}}^{tot}(k) = & (-\eta_{\alpha\gamma}\eta_{\dot{\alpha}\dot{\gamma}} + \frac{1}{2}\eta_{\alpha\dot{\gamma}}\eta_{\dot{\alpha}\gamma} + \frac{1}{2}\eta_{\alpha\dot{\alpha}}\eta_{\gamma\dot{\gamma}})k^2 + \eta_{\alpha\gamma}k_{\dot{\alpha}}k_{\dot{\gamma}} + \eta_{\dot{\alpha}\dot{\gamma}}k_{\alpha}k_{\gamma} \\ & - \frac{1}{2}(\eta_{\alpha\dot{\gamma}}k_{\dot{\alpha}}k_{\gamma} + \eta_{\dot{\alpha}\gamma}k_{\alpha}k_{\dot{\gamma}} + \eta_{\alpha\dot{\alpha}}k_{\gamma}k_{\dot{\gamma}} + \eta_{\gamma\dot{\gamma}}k_{\alpha}k_{\dot{\alpha}}). \end{aligned} \quad (35)$$

In summary, we have the following Lagrangian for the lower-rank tensor gauge fields:

$$\begin{aligned} \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_2' = & - \frac{1}{4}G_{\mu\nu}^a G_{\mu\nu}^a \\ & - \frac{1}{4}G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda}^a - \frac{1}{4}G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda}^a \\ & + \frac{1}{4}G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu}^a + \frac{1}{4}G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda}^a + \frac{1}{2}G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^a. \end{aligned} \quad (36)$$

Let us consider the equations of motion which follow from this Lagrangian for the vector gauge field  $A_\nu^a$ :

$$\begin{aligned} & \nabla_\mu^{ab} G_{\mu\nu}^b + \frac{1}{2} \nabla_\mu^{ab} (G_{\mu\nu,\lambda\lambda}^b - G_{\mu\lambda,\nu\lambda}^b - G_{\lambda\nu,\mu\lambda}^b) + g f^{acb} A_{\mu\lambda}^c G_{\mu\nu,\lambda}^b \\ & - \frac{1}{2} g f^{acb} (A_{\mu\lambda}^c G_{\mu\lambda,\nu}^b + A_{\lambda\mu}^c G_{\mu\nu,\lambda}^b + A_{\mu\nu}^c G_{\mu\lambda,\lambda}^b - A_{\lambda\lambda}^c G_{\mu\nu,\mu}^b - A_{\mu\lambda\lambda}^c G_{\mu\nu}^b + A_{\mu\lambda\nu}^c G_{\mu\lambda}^b) = 0 \end{aligned} \quad (37)$$

and for the second-rank tensor gauge field  $A_{\nu\lambda}^a$ :

$$\begin{aligned} & \nabla_\mu^{ab} G_{\mu\nu,\lambda}^b - \frac{1}{2} (\nabla_\mu^{ab} G_{\mu\lambda,\nu}^b + \nabla_\mu^{ab} G_{\lambda\nu,\mu}^b + \nabla_\lambda^{ab} G_{\mu\nu,\mu}^b + \eta_{\nu\lambda} \nabla_\mu^{ab} G_{\mu\rho,\rho}^b) \\ & + g f^{acb} A_{\mu\lambda}^c G_{\mu\nu}^b - \frac{1}{2} g f^{acb} (A_{\mu\nu}^c G_{\mu\lambda}^b + A_{\lambda\mu}^c G_{\mu\nu}^b + A_{\mu\mu}^c G_{\lambda\nu}^b + \eta_{\nu\lambda} A_{\mu\rho}^c G_{\mu\rho}^b) = 0. \end{aligned} \quad (38)$$

The variation of the action with respect to the third-rank gauge field  $A_{\nu\lambda\rho}^a$  will give the equations

$$\eta_{\lambda\rho} \nabla_\mu^{ab} G_{\mu\nu}^b - \frac{1}{2} (\eta_{\nu\rho} \nabla_\mu^{ab} G_{\mu\lambda}^b + \eta_{\nu\lambda} \nabla_\mu^{ab} G_{\mu\rho}^b) + \frac{1}{2} (\nabla_\rho^{ab} G_{\nu\lambda}^b + \nabla_\lambda^{ab} G_{\nu\rho}^b) = 0. \quad (39)$$

Representing these system of equations in the form

$$\begin{aligned} & \partial_\mu F_{\mu\nu,\lambda}^a = j_\nu^a \\ & \partial_\mu F_{\mu\nu,\lambda}^a - \frac{1}{2} (\partial_\mu F_{\mu\lambda,\nu}^a + \partial_\mu F_{\lambda\nu,\mu}^a + \partial_\lambda F_{\mu\nu,\mu}^a + \eta_{\nu\lambda} \partial_\mu F_{\mu\rho,\rho}^a) = j_{\nu\lambda}^a \\ & \eta_{\lambda\rho} \partial_\mu F_{\mu\nu}^a - \frac{1}{2} (\eta_{\nu\rho} \partial_\mu F_{\mu\lambda}^a + \eta_{\nu\lambda} \partial_\mu F_{\mu\rho}^a) + \frac{1}{2} (\partial_\rho F_{\nu\lambda}^a + \partial_\lambda F_{\nu\rho}^a) = j_{\nu\lambda\rho}^a, \end{aligned} \quad (40)$$

where  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ ,  $F_{\mu\nu,\lambda}^a = \partial_\mu A_{\nu\lambda}^a - \partial_\nu A_{\mu\lambda}^a$ ,  $F_{\mu\nu,\lambda\rho}^a = \partial_\mu A_{\nu\lambda\rho}^a - \partial_\nu A_{\mu\lambda\rho}^a$ , we can find the corresponding conserved currents

$$\begin{aligned} j_\nu^a = & - g f^{abc} A_\mu^b G_{\mu\nu}^c - g f^{abc} \partial_\mu (A_\mu^b A_\nu^c) \\ & - \frac{1}{2} g f^{abc} A_\mu^b (G_{\mu\nu,\lambda\lambda}^c - G_{\mu\lambda,\nu\lambda}^c - G_{\lambda\nu,\mu\lambda}^c) - \frac{1}{2} \partial_\mu (I_{\mu\nu,\lambda\lambda}^a - I_{\mu\lambda,\nu\lambda}^a - I_{\lambda\nu,\mu\lambda}^a) \\ & - g f^{abc} A_{\mu\lambda}^b G_{\mu\nu,\lambda}^c + \frac{1}{2} g f^{abc} (A_{\mu\lambda}^b G_{\mu\lambda,\nu}^c + A_{\lambda\mu}^b G_{\mu\nu,\lambda}^c + A_{\mu\nu}^b G_{\mu\lambda,\lambda}^c - A_{\lambda\lambda}^b G_{\mu\nu,\mu}^c) \\ & - \frac{1}{2} g f^{abc} (A_{\mu\lambda\lambda}^b G_{\mu\nu}^c - A_{\mu\lambda\nu}^b G_{\mu\lambda}^c), \end{aligned} \quad (41)$$

where  $I_{\mu\nu,\lambda\rho}^a = g f^{abc} (A_\mu^b A_{\nu\lambda\rho}^c + A_{\mu\lambda}^b A_{\nu\rho}^c + A_{\mu\rho}^b A_{\nu\lambda}^c + A_{\mu\lambda\rho}^b A_\nu^c)$  and

$$\begin{aligned} j_{\nu\lambda}^a = & - g f^{abc} A_\mu^b G_{\mu\nu,\lambda}^c + \frac{1}{2} g f^{abc} (A_\mu^b G_{\mu\lambda,\nu}^c + A_\mu^b G_{\lambda\nu,\mu}^c + A_\lambda^b G_{\mu\nu,\mu}^c + \eta_{\nu\lambda} A_\mu^b G_{\mu\rho,\rho}^c) \\ & - g f^{abc} A_{\mu\lambda}^b G_{\mu\nu}^c + \frac{1}{2} g f^{abc} (A_{\mu\nu}^b G_{\mu\lambda}^c + A_{\lambda\mu}^b G_{\mu\nu}^c + A_{\mu\mu}^b G_{\lambda\nu}^c + \eta_{\nu\lambda} A_{\mu\rho}^b G_{\mu\rho}^c) \\ & - g f^{abc} \partial_\mu (A_\mu^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_\nu^c) + \frac{1}{2} g f^{abc} [\partial_\mu (A_\mu^b A_{\lambda\nu}^c + A_{\mu\nu}^b A_\lambda^c) + \partial_\mu (A_\lambda^b A_{\nu\mu}^c + A_{\lambda\mu}^b A_\nu^c) \\ & + \partial_\lambda (A_\mu^b A_{\nu\mu}^c + A_{\mu\mu}^b A_\nu^c) + \eta_{\nu\lambda} \partial_\mu (A_\mu^b A_{\rho\rho}^c + A_{\mu\rho}^b A_\rho^c)], \end{aligned} \quad (42)$$

$$\begin{aligned} j_{\nu\lambda\rho}^a = & - \eta_{\lambda\rho} g f^{abc} A_\mu^b G_{\mu\nu}^c + \frac{1}{2} g f^{abc} (\eta_{\nu\rho} A_\mu^b G_{\mu\lambda}^c + \eta_{\nu\lambda} A_\mu^b G_{\mu\rho}^c - A_\rho^b G_{\nu\lambda}^c - A_\lambda^b G_{\nu\rho}^c) \\ & - \eta_{\lambda\rho} g f^{abc} \partial_\mu (A_\mu^b A_\nu^c) + \frac{1}{2} g f^{abc} [\partial_\mu (\eta_{\nu\lambda} A_\mu^b A_\rho^c + \eta_{\nu\rho} A_\mu^b A_\lambda^c) - \partial_\lambda (A_\nu^b A_\rho^c) - \partial_\rho (A_\nu^b A_\lambda^c)]. \end{aligned} \quad (43)$$

Thus

$$\begin{aligned}\partial_\nu j_\nu^a &= 0, \\ \partial_\nu j_{\nu\lambda}^a &= 0, \quad \partial_\lambda j_{\nu\lambda}^a = 0, \\ \partial_\nu j_{\nu\lambda\rho}^a &= 0, \quad \partial_\lambda j_{\nu\lambda\rho}^a = 0, \quad \partial_\rho j_{\nu\lambda\rho}^a = 0,\end{aligned}\tag{44}$$

because, as we demonstrated above, the partial derivatives of the l.h.s. of the equations (40) are equal to zero (see equations (32) and also equations (61)).

At the linearized level, when the gauge coupling constant  $g$  is equal to zero, the equations of motion (38) for the second-rank tensor gauge fields will take the form

$$\begin{aligned}\partial^2(A_{\nu\lambda}^a - \frac{1}{2}A_{\lambda\nu}^a) - \partial_\nu\partial_\mu(A_{\mu\lambda}^a - \frac{1}{2}A_{\lambda\mu}^a) &- \partial_\lambda\partial_\mu(A_{\nu\mu}^a - \frac{1}{2}A_{\mu\nu}^a) + \partial_\nu\partial_\lambda(A_{\mu\mu}^a - \frac{1}{2}A_{\mu\mu}^a) \\ &+ \frac{1}{2}\eta_{\nu\lambda}(\partial_\mu\partial_\rho A_{\mu\rho}^a - \partial^2 A_{\mu\mu}^a) = 0\end{aligned}\tag{45}$$

and, as we shall see below, they describe the propagation of massless particles of spin 2 and spin 0. First of all it is also easy to see that for the symmetric part of the tensor gauge field  $(A_{\nu\lambda}^a + A_{\lambda\nu}^a)/2$  our equation reduces to the well known Fierz-Pauli-Schwinger-Chang-Singh-Hagen-Fronsdal equation

$$\partial^2 A_{\nu\lambda} - \partial_\nu\partial_\mu A_{\mu\lambda} - \partial_\lambda\partial_\mu A_{\mu\nu} + \partial_\nu\partial_\lambda A_{\mu\mu} + \eta_{\nu\lambda}(\partial_\mu\partial_\rho A_{\mu\rho} - \partial^2 A_{\mu\mu}) = 0,\tag{46}$$

which describes the propagation of massless tensor boson with two physical polarizations, the  $s = \pm 2$  helicity states. For the antisymmetric part  $(A_{\nu\lambda}^a - A_{\lambda\nu}^a)/2$  the equation reduces to the form

$$\partial^2 A_{\nu\lambda} - \partial_\nu\partial_\mu A_{\mu\lambda} + \partial_\lambda\partial_\mu A_{\mu\nu} = 0\tag{47}$$

and describes the propagation of massless scalar boson with one physical polarization, the  $s = 0$  helicity state.

Alternatively, we can find out propagating degrees of freedom directly equation (45) in a particular gauge. Indeed, taking the trace of the equation (45) we shall get

$$\partial_\mu\partial_\rho A_{\mu\rho}^a - \partial^2 A_{\rho\rho}^a = 0,\tag{48}$$

and the equation (45) takes the form

$$\partial^2(A_{\nu\lambda}^a - \frac{1}{2}A_{\lambda\nu}^a) - \partial_\nu\partial_\mu(A_{\mu\lambda}^a - \frac{1}{2}A_{\lambda\mu}^a) - \partial_\lambda\partial_\mu(A_{\nu\mu}^a - \frac{1}{2}A_{\mu\nu}^a) + \frac{1}{2}\partial_\nu\partial_\lambda A_{\mu\mu}^a = 0.\tag{49}$$

Using the gauge invariance (31) we can impose the Lorentz invariant supplementary conditions on the second-rank gauge fields  $A_{\mu\lambda}$ :  $\partial_\mu A_{\mu\lambda}^a = a_\lambda$ ,  $\partial_\lambda A_{\mu\lambda}^a = b_\mu$ , where  $a_\lambda$  and  $b_\mu$  are arbitrary functions, or one can suggest alternative supplementary conditions in which the quadratic form (33), (34), (35) is diagonal:

$$\partial_\mu A_{\mu\lambda}^a - \frac{1}{2}\partial_\lambda A_{\mu\mu}^a = 0, \quad \partial_\lambda A_{\mu\lambda}^a - \frac{1}{2}\partial_\mu A_{\lambda\lambda}^a = 0.\tag{50}$$

In this gauge the equation (49) has the form

$$\partial^2 A_{\nu\lambda}^a = 0\tag{51}$$

and in the momentum representation  $A_{\mu\nu}(x) = e_{\mu\nu}(k)e^{ikx}$  from equation (51) it follows that  $k^2 = 0$  and we have *massless particles*.

For the symmetric part of the tensor field  $A_{\mu\lambda}^a$  the supplementary conditions (50) are equivalent to the harmonic gauge

$$\partial_\mu(A_{\mu\lambda}^a + A_{\lambda\mu}^a) - \frac{1}{2}\partial_\lambda(A_{\mu\mu}^a + A_{\mu\mu}^a) = 0, \quad (52)$$

and the residual gauge transformations are defined by the gauge parameters  $\xi_\lambda^a + \eta_\lambda^a$  which should satisfy the equation

$$\partial^2(\xi_\lambda^a + \eta_\lambda^a) = 0. \quad (53)$$

Thus imposing the harmonic gauge (52) and using the residual gauge transformations (53) one can see that the number of propagating physical polarizations which are described by the symmetric part of the tensor field  $A_{\mu\lambda}^a$  are given by two helicity states  $s = \pm 2$  multiplied by the dimension of the group G ( $a=1, \dots, N$ ).

For the anisymmetric part of the tensor field  $A_{\mu\lambda}^a$  the supplementary conditions (50) are equivalent to the Lorentz gauge

$$\partial_\mu(A_{\mu\lambda}^a - A_{\lambda\mu}^a) = 0 \quad (54)$$

and together with the equation of motion they describe the propagation of one physical polarization of helicity  $s = 0$  multiplied by the dimension of the group G ( $a=1, \dots, N$ ).

Thus we have seen that the extended gauge symmetry (31) with eight gauge parameters is sufficient to exclude all negative norm polarizations from the spectrum of the second-rank *nonsymmetric tensor gauge field*  $A_{\mu\lambda}$  which describes now the propagation of three physical modes of helicities  $\pm 2$  and 0.

In the gauge (50) we shall get

$$H_{\alpha\dot{\alpha}\gamma\dot{\gamma}}^{fix}(k) = (\eta_{\alpha\gamma}\eta_{\dot{\alpha}\dot{\gamma}} - \frac{1}{2}\eta_{\alpha\dot{\gamma}}\eta_{\dot{\alpha}\gamma} - \frac{1}{4}\eta_{\alpha\dot{\alpha}}\eta_{\gamma\dot{\gamma}})(-k^2)$$

and the propagator  $\Delta_{\gamma\dot{\gamma}\lambda\dot{\lambda}}(k)$  defined by the equation

$$H_{\alpha\dot{\alpha}\gamma\dot{\gamma}}^{fix}(k)\Delta_{\gamma\dot{\gamma}\lambda\dot{\lambda}}(k) = \eta_{\alpha\lambda}\eta_{\dot{\alpha}\dot{\lambda}},$$

will take the form

$$\Delta_{\gamma\dot{\gamma}\lambda\dot{\lambda}}(k) = -\frac{4\eta_{\gamma\lambda}\eta_{\dot{\gamma}\dot{\lambda}} + 2\eta_{\gamma\dot{\lambda}}\eta_{\dot{\gamma}\lambda} - 3\eta_{\gamma\dot{\gamma}}\eta_{\lambda\dot{\lambda}}}{3(k^2 - i\varepsilon)}. \quad (55)$$

The corresponding residue can be represented as a sum

$$\begin{aligned} \frac{4\eta_{\gamma\lambda}\eta_{\dot{\gamma}\dot{\lambda}} + 2\eta_{\gamma\dot{\lambda}}\eta_{\dot{\gamma}\lambda} - 3\eta_{\gamma\dot{\gamma}}\eta_{\lambda\dot{\lambda}}}{3} = & + (\eta_{\gamma\lambda}\eta_{\dot{\gamma}\dot{\lambda}} + \eta_{\gamma\dot{\lambda}}\eta_{\dot{\gamma}\lambda} - \eta_{\gamma\dot{\gamma}}\eta_{\lambda\dot{\lambda}}) + \\ & + \frac{1}{3}(\eta_{\gamma\lambda}\eta_{\dot{\gamma}\dot{\lambda}} - \eta_{\gamma\dot{\lambda}}\eta_{\dot{\gamma}\lambda}). \end{aligned} \quad (56)$$

The first term describes the  $s = \pm 2$  helicity states and is represented by the symmetric part of the polarization tensor  $e_{\mu\lambda}$ , the second term describes  $s = 0$  helicity state and is represented by its antisymmetric part. Indeed, for the massless case, when  $k_\mu$  is aligned

along the third axis,  $k_\mu = (k, 0, 0, k)$ , we have two independent polarizations of the helicity-2 particle:

$$e_{\mu\lambda}^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0, 0, & 0, 0 \\ 0, 1, & 0, 0 \\ 0, 0, & -1, 0 \\ 0, 0, & 0, 0 \end{pmatrix}, e_{\mu\lambda}^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0, 0, 0, 0 \\ 0, 0, 1, 0 \\ 0, 1, 0, 0 \\ 0, 0, 0, 0 \end{pmatrix}, \quad (57)$$

with the property that  $e_{\gamma\dot{\gamma}}^1 e_{\lambda\dot{\lambda}}^1 + e_{\alpha\dot{\alpha}}^2 e_{\gamma\dot{\gamma}}^2 \simeq \frac{1}{2}(\eta_{\gamma\lambda}\eta_{\dot{\gamma}\dot{\lambda}} + \eta_{\gamma\dot{\lambda}}\eta_{\dot{\gamma}\lambda} - \eta_{\gamma\dot{\gamma}}\eta_{\lambda\dot{\lambda}})$ . The symbol  $\simeq$  means that the equation holds up to longitudinal terms. The polarization tensor which characterizes the third spin-zero *axion* state has the form

$$e_{\mu\lambda}^A = \frac{1}{\sqrt{2}} \begin{pmatrix} 0, & 0, 0, 0 \\ 0, & 0, 1, 0 \\ 0, & -1, 0, 0 \\ 0, & 0, 0, 0 \end{pmatrix}, \quad (58)$$

and  $e_{\gamma\dot{\gamma}}^A e_{\lambda\dot{\lambda}}^A \simeq \frac{1}{2}(\eta_{\gamma\lambda}\eta_{\dot{\gamma}\dot{\lambda}} - \eta_{\gamma\dot{\lambda}}\eta_{\dot{\gamma}\lambda})$ . Thus the general second-rank tensor gauge field with 16 components  $A_{\mu\lambda}$  describes in this theory three physical propagating massless polarizations.

## 4 Enhanced Local Gauge Algebra

Let us consider now the symmetries of the remaining two terms in the full Lagrangian  $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}'_2$ . They have the form

$$-\frac{1}{4}G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda}^a + \frac{1}{2}G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^a.$$

The part which is quadratic in fields has the form

$$\begin{aligned} \mathcal{L}_2^{free} &= \frac{1}{2}A_\alpha^a \{ \quad + \quad (\eta_{\alpha\gamma}\partial^2 - \partial_\alpha\partial_\gamma)\eta_{\gamma'\gamma''} - \eta_{\alpha\gamma}\partial_{\gamma'}\partial_{\gamma''} \\ &\quad - \quad \frac{1}{2}(\eta_{\gamma\gamma''}\partial^2 - \partial_\gamma\partial_{\gamma''})\eta_{\alpha\gamma'} - \frac{1}{2}(\eta_{\gamma\gamma'}\partial^2 - \partial_\gamma\partial_{\gamma'})\eta_{\alpha\gamma''} \\ &\quad + \quad \frac{1}{2}\eta_{\gamma\gamma''}\partial_\alpha\partial_{\gamma'} + \frac{1}{2}\eta_{\gamma\gamma'}\partial_\alpha\partial_{\gamma''} \} A_{\gamma\gamma'\gamma''}^a \\ &= \frac{1}{2}A_\alpha^a H_{\alpha\gamma\gamma'\gamma''} A_{\gamma\gamma'\gamma''}^a, \end{aligned} \quad (59)$$

where the quadratic form in the momentum representation is

$$\begin{aligned} H_{\alpha\gamma\gamma'\gamma''}(k) &= - \quad (\eta_{\alpha\gamma}k^2 - k_\alpha k_\gamma)\eta_{\gamma'\gamma''} + \eta_{\alpha\gamma}k_{\gamma'}k_{\gamma''} \\ &\quad + \quad \frac{1}{2}(\eta_{\gamma\gamma''}k^2 - k_\gamma k_{\gamma''})\eta_{\alpha\gamma'} - \frac{1}{2}\eta_{\gamma\gamma''}k_\alpha k_{\gamma'} \\ &\quad + \quad \frac{1}{2}(\eta_{\gamma\gamma'}k^2 - k_\gamma k_{\gamma'})\eta_{\alpha\gamma''} - \frac{1}{2}\eta_{\gamma\gamma'}k_\alpha k_{\gamma''}. \end{aligned} \quad (60)$$

It is useful to represent it in three equivalent forms

$$\begin{aligned}
H_{\alpha\gamma\gamma'\gamma''}(k) &= - H_{\alpha\gamma}\eta_{\gamma'\gamma''} + \frac{1}{2}H_{\gamma\gamma''}\eta_{\alpha\gamma'} + \frac{1}{2}H_{\gamma\gamma'}\eta_{\alpha\gamma''} \\
&+ \frac{1}{2}(\eta_{\alpha\gamma}k_{\gamma''} - \eta_{\gamma\gamma''}k_{\alpha})k_{\gamma'} \\
&+ \frac{1}{2}(\eta_{\alpha\gamma}k_{\gamma'} - \eta_{\gamma\gamma'}k_{\alpha})k_{\gamma''} \\
&= - H_{\gamma'\gamma''}\eta_{\alpha\gamma} + \frac{1}{2}H_{\alpha\gamma'}\eta_{\gamma\gamma''} + \frac{1}{2}H_{\alpha\gamma''}\eta_{\gamma\gamma'} \\
&+ \frac{1}{2}(\eta_{\gamma'\gamma''}k_{\alpha} - \eta_{\alpha\gamma'}k_{\gamma''})k_{\gamma} \\
&+ \frac{1}{2}(\eta_{\gamma'\gamma''}k_{\alpha} - \eta_{\alpha\gamma''}k_{\gamma'})k_{\gamma} \\
&= - H_{\alpha\gamma}\eta_{\gamma'\gamma''} + \frac{1}{2}H_{\alpha\gamma'}\eta_{\gamma\gamma''} + \frac{1}{2}H_{\alpha\gamma''}\eta_{\gamma\gamma'} \\
&+ \frac{1}{2}(\eta_{\alpha\gamma}k_{\gamma'} - \eta_{\alpha\gamma'}k_{\gamma})k_{\gamma''} \\
&+ \frac{1}{2}(\eta_{\alpha\gamma}k_{\gamma''} - \eta_{\alpha\gamma''}k_{\gamma})k_{\gamma'}.
\end{aligned}$$

As one can see all divergences are equal to zero

$$k_{\alpha}H_{\alpha\gamma\gamma'\gamma''}(k) = k_{\gamma}H_{\alpha\gamma\gamma'\gamma''}(k) = k_{\gamma'}H_{\alpha\gamma\gamma'\gamma''}(k) = k_{\gamma''}H_{\alpha\gamma\gamma'\gamma''}(k) = 0. \quad (61)$$

This result means that the quadratic part of the full Lagrangian  $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}'_2$  is invariant under the following local gauge transformations

$$\begin{aligned}
\tilde{\delta}_{\eta}A_{\mu}^a &= \partial_{\mu}\eta^a + \dots \\
(II) \quad \tilde{\delta}_{\eta}A_{\mu\nu}^a &= \partial_{\nu}\eta_{\mu}^a + \dots, \\
\tilde{\delta}_{\eta}A_{\mu\nu\lambda}^a &= \partial_{\nu}\eta_{\mu\lambda}^a + \partial_{\lambda}\eta_{\mu\nu}^a + \dots \\
&\dots\dots\dots
\end{aligned} \quad (62)$$

in addition to the initial local gauge transformations (3)

$$\begin{aligned}
\delta_{\xi}A_{\mu}^a &= \partial_{\mu}\xi^a + \dots \\
(I) \quad \delta_{\xi}A_{\mu\nu}^a &= \partial_{\mu}\xi_{\nu}^a + \dots \\
\delta_{\xi}A_{\mu\nu\lambda}^a &= \partial_{\mu}\xi_{\nu\lambda}^a + \dots
\end{aligned} \quad (63)$$

It is important to know how the transformation (62) looks like when the gauge coupling constant is not equal to zero. The existence of the full transformation is guaranteed by the conservation of the corresponding currents (41), (42) and (43). At the moment we can only guess the full form of the second local gauge transformation requiring the closure of the corresponding algebra. The extension we have found has the form [2]:

$$\begin{aligned}
\tilde{\delta}_{\eta}A_{\mu}^a &= (\delta^{ab}\partial_{\mu} + gf^{acb}A_{\mu}^c)\eta^b, \\
(II) \quad \tilde{\delta}_{\eta}A_{\mu\nu}^a &= (\delta^{ab}\partial_{\nu} + gf^{acb}A_{\nu}^c)\eta_{\mu}^b + gf^{acb}A_{\mu\nu}^c\eta^b, \\
\tilde{\delta}_{\eta}A_{\mu\nu\lambda}^a &= (\delta^{ab}\partial_{\nu} + gf^{acb}A_{\nu}^c)\eta_{\mu\lambda}^b + (\delta^{ab}\partial_{\lambda} + gf^{acb}A_{\lambda}^c)\eta_{\mu\nu}^b \\
&\quad + gf^{acb}(A_{\mu\nu}^c\eta_{\lambda}^b + A_{\mu\lambda}^c\eta_{\nu}^b + A_{\mu\nu\lambda}^c\eta^b) \\
&\dots\dots\dots
\end{aligned} \quad (64)$$

and forms a closed algebraic structure. The composition law of the gauge parameters  $\{\eta, \eta_\nu, \eta_{\nu\lambda}, \dots\}$  is the same as in (7).

## 5 Interaction Vertices

We are interested now to analyze the interaction properties of the tensor gauge bosons prescribed by the gauge principle. The interaction of the Yang-Mills vector bosons with the charged tensor gauge bosons is described by the Lagrangian (36)  $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}'_2$ . Let us first consider three-particle interaction vertices - VTT. Explicitly three-linear terms of the Lagrangian  $\mathcal{L}_2$  have the form:

$$\begin{aligned} \mathcal{L}_2^{cubic} = & - \frac{1}{2} g f^{abc} (\partial_\mu A_{\nu\lambda}^a - \partial_\nu A_{\mu\lambda}^a) (A_\mu^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_\nu^c) \\ & - \frac{1}{4} g f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) 2A_{\mu\lambda}^b A_{\nu\lambda}^c, \end{aligned} \quad (65)$$

and in addition to the standard Yang-Mills VVV three vector boson interaction vertex

$$\mathcal{L}_1^{cubic} = -\frac{1}{2} g f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) A_\mu^b A_\nu^c,$$

which in the momentum representation has the form

$$\mathcal{V}_{\alpha\beta\gamma}^{abc}(k, p, q) = -ig f^{abc} [\eta_{\alpha\beta}(p-k)_\gamma + \eta_{\alpha\gamma}(k-q)_\beta + \eta_{\beta\gamma}(q-p)_\alpha] = -ig f^{abc} F_{\alpha\beta\gamma}(k, p, q), \quad (66)$$

we have a new three-particle interaction vertex of one vector boson and two tensor gauge bosons - VTT. In momentum space it has the form

$$\mathcal{V}_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}^{abc}(k, p, q) = -3ig f^{abc} [\eta_{\alpha\beta}(p-k)_\gamma + \eta_{\alpha\gamma}(k-q)_\beta + \eta_{\beta\gamma}(q-p)_\alpha] \eta_{\dot{\alpha}\dot{\gamma}}. \quad (67)$$

Notice that two parts in (65), which came from different terms of the Lagrangian  $\mathcal{L}_2$ , combine into the VVV vertex and the tensor  $3\eta_{\dot{\alpha}\dot{\gamma}}$ . It is convenient to represent the vertex in the form

$$\mathcal{V}_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}^{abc}(k, p, q) = -3ig f^{abc} F_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}(k, p, q). \quad (68)$$

We have also a three-particle interaction vertex VTT of one vector boson and two tensor gauge bosons in the second Lagrangian  $\mathcal{L}'_2$ . Explicitly the three-linear terms of Lagrangian  $\mathcal{L}'_2$  have the form:

$$\begin{aligned} \mathcal{L}'_2^{cubic} = & \frac{1}{2} g f^{abc} (\partial_\mu A_{\nu\lambda}^a - \partial_\nu A_{\mu\lambda}^a) (A_\mu^b A_{\lambda\nu}^c + A_{\mu\nu}^b A_\lambda^c) \\ & + \frac{1}{2} g f^{abc} (\partial_\mu A_{\nu\nu}^a - \partial_\nu A_{\mu\nu}^a) (A_\mu^b A_{\lambda\lambda}^c + A_{\mu\lambda}^b A_\lambda^c) \\ & + \frac{1}{2} g f^{abc} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) (A_{\mu\nu}^b A_{\lambda\lambda}^c + A_{\mu\lambda}^b A_{\lambda\nu}^c), \end{aligned} \quad (69)$$

so that in the momentum space we have

$$\mathcal{V}'_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}^{abc}(k, p, q) = \frac{3}{2} ig f^{abc} F'_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}(k, p, q)$$

$$\begin{aligned}
F'_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}(k, p, q) = & (p-k)_\gamma(\eta_{\alpha\dot{\gamma}}\eta_{\dot{\alpha}\beta} + \eta_{\alpha\dot{\alpha}}\eta_{\beta\dot{\gamma}}) \\
& + (k-q)_\beta(\eta_{\alpha\dot{\gamma}}\eta_{\dot{\alpha}\gamma} + \eta_{\alpha\dot{\alpha}}\eta_{\gamma\dot{\gamma}}) \\
& + (q-p)_\alpha(\eta_{\dot{\alpha}\gamma}\eta_{\beta\dot{\gamma}} + \eta_{\dot{\alpha}\beta}\eta_{\gamma\dot{\gamma}}) \\
& + (p-k)_{\dot{\alpha}}\eta_{\alpha\beta}\eta_{\gamma\dot{\gamma}} + (p-k)_{\dot{\gamma}}\eta_{\alpha\beta}\eta_{\dot{\alpha}\gamma} \\
& + (k-q)_{\dot{\alpha}}\eta_{\alpha\gamma}\eta_{\beta\dot{\gamma}} + (k-q)_{\dot{\gamma}}\eta_{\alpha\gamma}\eta_{\dot{\alpha}\beta} \\
& + (q-p)_{\dot{\alpha}}\eta_{\beta\gamma}\eta_{\alpha\dot{\gamma}} + (q-p)_{\dot{\gamma}}\eta_{\alpha\dot{\alpha}}\eta_{\beta\gamma}.
\end{aligned} \tag{70}$$

Collecting two terms of the three-point vertex VTT together we shall get

$$\mathcal{V}_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}^{tot\ abc}(k, p, q) = \mathcal{V}_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}^{abc}(k, p, q) + \mathcal{V}'_{\alpha\dot{\alpha}\beta\gamma\dot{\gamma}}(k, p, q). \tag{71}$$

Let us consider now four-particle interaction terms of the Lagrangian  $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}'_2$ . We have the standard four vector boson interaction vertex VVVV

$$\begin{aligned}
\mathcal{V}_{\alpha\beta\gamma\delta}^{abcd}(k, p, q, r) = & -g^2 f^{lac} f^{lbd}(\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma}) \\
& -g^2 f^{lad} f^{lbc}(\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{\alpha\gamma}\eta_{\beta\delta}) \\
& -g^2 f^{lab} f^{lcd}(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma})
\end{aligned} \tag{72}$$

and a new interaction of two vector and two tensor gauge bosons - the VVTT vertex,

$$\begin{aligned}
\mathcal{L}_2^{quartic} = & -\frac{1}{4}g^2 f^{abc} f^{ab\dot{c}}(A_\mu^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_\nu^c)(A_\mu^{\dot{b}} A_{\nu\lambda}^{\dot{c}} + A_{\mu\lambda}^{\dot{b}} A_\nu^{\dot{c}}) \\
& -\frac{1}{2}g^2 f^{abc} f^{ab\dot{c}} A_\mu^b A_\nu^c A_{\mu\lambda}^{\dot{b}} A_{\nu\lambda}^{\dot{c}},
\end{aligned} \tag{73}$$

which in the momentum space will take the form

$$\begin{aligned}
\mathcal{V}_{\alpha\beta\gamma\dot{\gamma}\delta\dot{\delta}}^{abcd}(k, p, q, r) = & -6g^2 f^{lac} f^{lbd}(\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma})\eta_{\dot{\gamma}\dot{\delta}} \\
& -6g^2 f^{lad} f^{lbc}(\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{\alpha\gamma}\eta_{\beta\delta})\eta_{\dot{\gamma}\dot{\delta}} \\
& -6g^2 f^{lab} f^{lcd}(\eta_{\alpha\gamma}\eta_{\beta\delta} - \eta_{\alpha\delta}\eta_{\beta\gamma})\eta_{\dot{\gamma}\dot{\delta}}.
\end{aligned} \tag{74}$$

The second part of the vertex VVTT comes from the Lagrangian  $\mathcal{L}'_2$ :

$$\begin{aligned}
\mathcal{L}'_2^{quartic} = & +\frac{1}{4}g^2 f^{abc} f^{ab\dot{c}}(A_\mu^b A_{\nu\lambda}^c + A_{\mu\lambda}^b A_\nu^c)(A_\mu^{\dot{b}} A_{\lambda\nu}^{\dot{c}} + A_{\mu\nu}^{\dot{b}} A_\lambda^{\dot{c}}) \\
& +\frac{1}{4}g^2 f^{abc} f^{ab\dot{c}}(A_\mu^b A_{\nu\nu}^c + A_{\mu\nu}^b A_\nu^c)(A_\mu^{\dot{b}} A_{\lambda\lambda}^{\dot{c}} + A_{\mu\lambda}^{\dot{b}} A_\lambda^{\dot{c}}) \\
& +\frac{1}{2}g^2 f^{abc} f^{ab\dot{c}} A_\mu^b A_\nu^c (A_{\mu\nu}^{\dot{b}} A_{\lambda\lambda}^{\dot{c}} + A_{\mu\lambda}^{\dot{b}} A_{\lambda\nu}^{\dot{c}}),
\end{aligned} \tag{75}$$

which in the momentum representation will take the form

$$\begin{aligned}
\mathcal{V}'_{\alpha\beta\gamma\dot{\gamma}\delta\dot{\delta}}^{abcd}(k, p, q, r) = & 3g^2 f^{lac} f^{lbd} [ + \eta_{\alpha\beta}(\eta_{\gamma\delta}\eta_{\dot{\gamma}\dot{\delta}} + \eta_{\gamma\dot{\gamma}}\eta_{\delta\dot{\delta}}) \\
& - \eta_{\beta\gamma}(\eta_{\alpha\delta}\eta_{\dot{\gamma}\dot{\delta}} + \eta_{\alpha\dot{\gamma}}\eta_{\delta\dot{\delta}}) \\
& - \eta_{\alpha\delta}(\eta_{\beta\dot{\gamma}}\eta_{\gamma\dot{\delta}} + \eta_{\beta\dot{\delta}}\eta_{\gamma\dot{\gamma}}) \\
& + \eta_{\gamma\delta}(\eta_{\alpha\dot{\delta}}\eta_{\beta\dot{\gamma}} + \eta_{\alpha\dot{\gamma}}\eta_{\beta\dot{\delta}}) ] \\
& 3g^2 f^{lad} f^{lbc} [ + \eta_{\alpha\beta}(\eta_{\gamma\delta}\eta_{\dot{\gamma}\dot{\delta}} + \eta_{\gamma\dot{\gamma}}\eta_{\delta\dot{\delta}})
\end{aligned}$$



$$\begin{aligned}
& - \eta_{\alpha\gamma}(\eta_{\beta\delta}\eta_{\gamma\delta} + \eta_{\beta\gamma}\eta_{\delta\delta}) \\
& - \eta_{\beta\delta}(\eta_{\alpha\gamma}\eta_{\gamma\delta} + \eta_{\alpha\delta}\eta_{\gamma\gamma}) \\
& + \eta_{\gamma\delta}(\eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma})] \\
3g^2 f^{lab} f^{lcd} [ & + \eta_{\alpha\gamma}(\eta_{\beta\gamma}\eta_{\delta\delta} + \eta_{\beta\delta}\eta_{\delta\gamma}) \\
& - \eta_{\beta\gamma}(\eta_{\alpha\gamma}\eta_{\delta\delta} + \eta_{\alpha\delta}\eta_{\delta\gamma}) \\
& - \eta_{\alpha\delta}(\eta_{\beta\delta}\eta_{\gamma\gamma} + \eta_{\beta\gamma}\eta_{\gamma\delta}) \\
& + \eta_{\beta\delta}(\eta_{\alpha\delta}\eta_{\gamma\gamma} + \eta_{\alpha\gamma}\eta_{\gamma\delta})]. \tag{76}
\end{aligned}$$

The total vertex is

$$\mathcal{V}_{\alpha\beta\gamma\delta}^{tot\ abcd}(k, p, q, r) = \mathcal{V}_{\alpha\beta\gamma\delta}^{abcd}(k, p, q, r) + \mathcal{V}'_{\alpha\beta\gamma\delta\delta}{}^{abcd}(k, p, q, r). \tag{77}$$

## 6 Third-Rank Tensor Gauge Fields

The Lagrangian  $\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}'_2$  contains the third-rank gauge fields  $A_{\mu\nu\lambda}^a$ , but without the corresponding kinetic term. In order to make the fields  $A_{\mu\nu\lambda}^a$  dynamical we have added the corresponding Lagrangian  $\mathcal{L}_3$  presented at the second line of the formula (14). But again the Lagrangian  $\mathcal{L}_3$  is not the most general invariant which can be constructed from the corresponding field strength tensors. There are seven Lorentz invariant quadratic forms which form the second invariant Lagrangian  $\mathcal{L}'_3$  so that at this level the total Lagrangian is a sum

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}'_2 + \mathcal{L}_3 + \mathcal{L}'_3 + \dots$$

Indeed, the Lagrangian  $\mathcal{L}_3$  has the form (14):

$$\mathcal{L}_3 = -\frac{1}{4}G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - \frac{1}{8}G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a - \frac{1}{2}G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda\rho\rho}^a - \frac{1}{8}G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda\rho\rho}^a, \tag{78}$$

where the field strength tensors (9) are

$$\begin{aligned}
G_{\mu\nu,\lambda\rho\sigma}^a = & \partial_\mu A_{\nu\lambda\rho\sigma}^a - \partial_\nu A_{\mu\lambda\rho\sigma}^a + gf^{abc}\{ A_\mu^b A_{\nu\lambda\rho\sigma}^c + A_{\mu\lambda}^b A_{\nu\rho\sigma}^c + A_{\mu\rho}^b A_{\nu\lambda\sigma}^c + A_{\mu\sigma}^b A_{\nu\lambda\rho}^c + \\
& + A_{\mu\lambda\rho}^b A_{\nu\sigma}^c + A_{\mu\lambda\sigma}^b A_{\nu\rho}^c + A_{\mu\rho\sigma}^b A_{\nu\lambda}^c + A_{\mu\lambda\rho\sigma}^b A_\nu^c \}
\end{aligned}$$

and

$$\begin{aligned}
G_{\mu\nu,\lambda\rho\sigma\delta}^a = & \partial_\mu A_{\nu\lambda\rho\sigma\delta}^a - \partial_\nu A_{\mu\lambda\rho\sigma\delta}^a + gf^{abc}\{ A_\mu^b A_{\nu\lambda\rho\sigma\delta}^c + \sum_{\lambda \leftrightarrow \rho, \sigma, \delta} A_{\mu\lambda}^b A_{\nu\rho\sigma\delta}^c + \\
& + \sum_{\lambda, \rho \leftrightarrow \sigma, \delta} A_{\mu\lambda\rho}^b A_{\nu\sigma\delta}^c + \sum_{\lambda, \rho, \sigma \leftrightarrow \delta} A_{\mu\lambda\rho\sigma}^b A_{\nu\delta}^c + A_{\mu\lambda\rho\sigma\delta}^b A_\nu^c \}.
\end{aligned}$$

The terms in parenthesis are symmetric over  $\lambda\rho\sigma$  and  $\lambda\rho\sigma\delta$  respectively. The Lagrangian  $\mathcal{L}_3$  is invariant with respect to the extended gauge transformations (3) of the low-rank gauge fields  $A_\mu, A_{\mu\nu}, A_{\mu\nu\lambda}$  and of the fourth-rank gauge field (5)

$$\begin{aligned}
\delta_\xi A_{\mu\nu\lambda\rho} = & \partial_\mu \xi_{\nu\lambda\rho} - ig[A_\mu, \xi_{\nu\lambda\rho}] - ig[A_{\mu\nu}, \xi_{\lambda\rho}] - ig[A_{\mu\lambda}, \xi_{\nu\rho}] - ig[A_{\mu\rho}, \xi_{\nu\lambda}] - \\
& - ig[A_{\mu\nu\lambda}, \xi_\rho] - ig[A_{\mu\nu\rho}, \xi_\lambda] - ig[A_{\mu\lambda\rho}, \xi_\nu] - ig[A_{\mu\nu\lambda\rho}, \xi]
\end{aligned}$$

and also of the fifth-rank tensor gauge field (5)

$$\begin{aligned}\delta_\xi A_{\mu\nu\lambda\rho\sigma} = & \partial_\mu \xi_{\nu\lambda\rho\sigma} - ig[A_\mu, \xi_{\nu\lambda\rho\sigma}] - ig \sum_{\nu \leftrightarrow \lambda\rho\sigma} [A_{\mu\nu}, \xi_{\lambda\rho\sigma}] - \\ & - ig \sum_{\nu\lambda \leftrightarrow \rho\sigma} [A_{\mu\nu\lambda}, \xi_{\rho\sigma}] - ig \sum_{\nu\lambda\rho \leftrightarrow \sigma} [A_{\mu\nu\lambda\rho}, \xi_\sigma] - ig[A_{\mu\nu\lambda\rho}, \xi],\end{aligned}$$

where the gauge parameters  $\xi_{\nu\lambda\rho}$  and  $\xi_{\nu\lambda\rho\sigma}$  are totally symmetric rank-3 and rank-4 tensors. The extended gauge transformation of the higher-rank tensor gauge fields induces the gauge transformation of the fields strengths of the form (11)

$$\begin{aligned}\delta G_{\mu\nu,\lambda\rho\sigma}^a = g f^{abc} ( & G_{\mu\nu,\lambda\rho\sigma}^b \xi^c + G_{\mu\nu,\lambda\rho}^b \xi_\sigma^c + G_{\mu\nu,\lambda\sigma}^b \xi_\rho^c + G_{\mu\nu,\rho\sigma}^b \xi_\lambda^c + \\ & + G_{\mu\nu,\lambda}^b \xi_{\rho\sigma}^c + G_{\mu\nu,\rho}^b \xi_{\lambda\sigma}^c + G_{\mu\nu,\sigma}^b \xi_{\lambda\rho}^c + G_{\mu\nu}^b \xi_{\lambda\rho\sigma}^c )\end{aligned}$$

and

$$\begin{aligned}\delta G_{\mu\nu,\lambda\rho\sigma\delta}^a = g f^{abc} ( & G_{\mu\nu,\lambda\rho\sigma\delta}^b \xi^c + \sum_{\lambda\rho,\sigma \leftrightarrow \delta} G_{\mu\nu,\lambda\rho\sigma}^b \xi_\delta^c + \\ & + \sum_{\lambda\rho \leftrightarrow \sigma,\delta} G_{\mu\nu,\lambda\rho}^b \xi_{\sigma\delta}^c + \sum_{\lambda \leftrightarrow \rho,\sigma,\delta} G_{\mu\nu,\lambda}^b \xi_{\rho\sigma\delta}^c + G_{\mu\nu}^b \xi_{\lambda\rho\sigma\delta}^c ).\end{aligned}$$

Using the above homogeneous transformations for the field strength tensors one can demonstrate the invariance of the Lagrangian  $\mathcal{L}_3$  with respect to the extended gauge transformations (10), (79) and (79) (see reference [3] for details).

Our purpose now is to present a second invariant Lagrangian which can be constructed in terms of the above field strength tensors. Let us consider the following seven Lorentz invariant quadratic forms which can be constructed by the corresponding field strength tensors

$$\begin{aligned}G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a, & G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a, & G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a, & G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a, \\ G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a, & G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a, & G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a.\end{aligned}\quad (79)$$

Calculating the variation of each of these terms with respect to the gauge transformation (10), (79) and (79) one can get convinced that the particular linear combination

$$\begin{aligned}\mathcal{L}'_3 = & \frac{1}{4} G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a + \frac{1}{4} G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a + \frac{1}{4} G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a \\ & + \frac{1}{4} G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a + \frac{1}{2} G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a + \frac{1}{4} G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a + \frac{1}{4} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a\end{aligned}\quad (80)$$

forms an invariant Lagrangian. Indeed, the variation of the first term is

$$\delta_\xi G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a = 2g f^{abc} G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu}^b \xi_\rho^c + 2g f^{abc} G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\rho}^b \xi_\nu^c + 2g f^{abc} G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda}^b \xi_{\nu\rho}^c,$$

of the second term is

$$\delta_\xi G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a = 2g f^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho}^b \xi_\lambda^c + 2g f^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\lambda}^b \xi_\rho^c + 2g f^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\rho}^b \xi_{\rho\lambda}^c,$$

of the third term is

$$\begin{aligned}\delta_\xi G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a = & 2g f^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho}^b \xi_\rho^c + g f^{abc} G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda}^b \xi_{\rho\rho}^c + g f^{abc} G_{\mu\lambda,\rho\rho}^a G_{\mu\nu,\nu}^b \xi_\lambda^c + \\ & + g f^{abc} G_{\mu\lambda,\rho\rho}^a G_{\mu\nu,\lambda}^b \xi_\nu^c + g f^{abc} G_{\mu\lambda,\rho\rho}^a G_{\mu\nu}^b \xi_{\rho\lambda}^c,\end{aligned}$$

of the forth term is

$$\begin{aligned} \delta_\xi G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a = & g f^{abc} G_{\mu\lambda,\nu\rho\rho}^a G_{\mu\nu,\lambda}^b \xi_\rho^c + 2g f^{abc} G_{\mu\lambda,\nu\rho}^a G_{\mu\nu,\lambda}^b \xi_\rho^c + g f^{abc} G_{\mu\lambda,\rho\rho}^a G_{\mu\nu,\lambda}^b \xi_\nu^c + \\ & + g f^{abc} G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu}^b \xi_{\rho\rho}^c + 2g f^{abc} G_{\mu\nu,\lambda}^a G_{\mu\lambda,\rho}^b \xi_{\nu\rho}^c + g f^{abc} G_{\mu\nu,\lambda}^a G_{\mu\lambda}^b \xi_{\nu\rho\rho}^c, \end{aligned}$$

of the fifth term is

$$\begin{aligned} \delta_\xi G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a = & g f^{abc} G_{\mu\rho,\nu\lambda\rho}^a G_{\mu\nu,\lambda}^b \xi_\lambda^c + g f^{abc} G_{\mu\rho,\nu\lambda}^a G_{\mu\nu,\lambda}^b \xi_\rho^c + g f^{abc} G_{\mu\rho,\nu\rho}^a G_{\mu\nu,\lambda}^b \xi_\lambda^c + g f^{abc} G_{\mu\rho,\lambda\rho}^a G_{\mu\nu,\lambda}^b \xi_\nu^c + \\ & + g f^{abc} G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu}^b \xi_{\lambda\rho}^c + g f^{abc} G_{\mu\nu,\lambda}^a G_{\mu\rho,\lambda}^b \xi_{\nu\rho}^c + g f^{abc} G_{\mu\nu,\lambda}^a G_{\mu\rho,\rho}^b \xi_{\nu\lambda}^c + g f^{abc} G_{\mu\nu,\lambda}^a G_{\mu\rho}^b \xi_{\nu\lambda\rho}^c, \end{aligned}$$

of the sixth term is

$$\begin{aligned} \delta_\xi G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a = & g f^{abc} G_{\mu\lambda,\lambda\rho\rho}^a G_{\mu\nu,\nu}^b \xi_\nu^c + 2g f^{abc} G_{\mu\lambda,\lambda\rho}^a G_{\mu\nu,\nu}^b \xi_\rho^c + g f^{abc} G_{\mu\lambda,\rho\rho}^a G_{\mu\nu,\nu}^b \xi_\lambda^c + g f^{abc} G_{\mu\lambda,\lambda}^a G_{\mu\nu,\nu}^b \xi_{\rho\rho}^c + \\ & + 2g f^{abc} G_{\mu\lambda,\rho}^a G_{\mu\nu,\nu}^b \xi_{\lambda\rho}^c + g f^{abc} G_{\mu\nu,\nu}^a G_{\mu\lambda}^b \xi_{\lambda\rho\rho}^c \end{aligned}$$

and finally of the seventh term is

$$\begin{aligned} \delta_\xi G_{\mu,\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a = & 2g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^b \xi_\rho^c + g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\nu\rho\rho}^b \xi_\lambda^c + g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\lambda\rho\rho}^b \xi_\nu^c + g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda}^b \xi_{\rho\rho}^c + \\ & 2g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\nu\rho}^b \xi_{\lambda\rho}^c + 2g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\lambda\rho}^b \xi_{\nu\rho}^c + g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\rho\rho}^b \xi_{\nu\lambda}^c + \\ & g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\nu}^b \xi_{\lambda\rho\rho}^c + g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\lambda}^b \xi_{\nu\rho\rho}^c + 2g f^{abc} G_{\mu\nu}^a G_{\mu\lambda,\rho}^b \xi_{\nu\lambda\rho}^c + g f^{abc} G_{\mu\nu}^a G_{\mu\lambda}^b \xi_{\nu\lambda\rho\rho}^c. \end{aligned}$$

Some of the terms here are equal to zero, like:  $g f^{abc} G_{\mu\nu,\lambda}^a G_{\mu\rho,\lambda}^b \xi_{\nu\rho}^c$ ,  $g f^{abc} G_{\mu\lambda,\lambda}^a G_{\mu\nu,\nu}^b \xi_{\rho\rho}^c$  and  $g f^{abc} G_{\mu\nu}^a G_{\mu\lambda}^b \xi_{\nu\lambda\rho\rho}^c$ . Amazingly all nonzero terms cancel each other.

In summary, we have the following Lagrangian for the third-rank gauge field  $A_{\mu\nu\lambda}^a$ :

$$\begin{aligned} \mathcal{L}_3 + c\mathcal{L}'_3 = & -\frac{1}{4}G_{\mu\nu,\lambda\rho}^a G_{\mu\nu,\lambda\rho}^a - \frac{1}{8}G_{\mu\nu,\lambda\lambda}^a G_{\mu\nu,\rho\rho}^a - \frac{1}{2}G_{\mu\nu,\lambda}^a G_{\mu\nu,\lambda\rho\rho}^a - \frac{1}{8}G_{\mu\nu}^a G_{\mu\nu,\lambda\lambda\rho\rho}^a + \\ & + \frac{c}{4}G_{\mu\nu,\lambda\rho}^a G_{\mu\lambda,\nu\rho}^a + \frac{c}{4}G_{\mu\nu,\nu\lambda}^a G_{\mu\rho,\rho\lambda}^a + \frac{c}{4}G_{\mu\nu,\nu\lambda}^a G_{\mu\lambda,\rho\rho}^a + \\ & + \frac{c}{4}G_{\mu\nu,\lambda}^a G_{\mu\lambda,\nu\rho\rho}^a + \frac{c}{2}G_{\mu\nu,\lambda}^a G_{\mu\rho,\nu\lambda\rho}^a + \frac{c}{4}G_{\mu\nu,\nu}^a G_{\mu\lambda,\lambda\rho\rho}^a + \frac{c}{4}G_{\mu\nu}^a G_{\mu\lambda,\nu\lambda\rho\rho}^a, \end{aligned} \quad (81)$$

where c is an arbitrary constant. As one can now be convinced this Lagrangian coincides with the Lagrangian (25) when s=2.

In summary at every "level" s we have two invariant quadratic forms, they represent a general Lagrangian at level s. The total Lagrangian is a linear sum of the two Lagrangians  $\mathcal{L}_s + c\mathcal{L}'_s$  which are given by formulas (12) and (25).

## 7 Conclusion

The transformations considered in the previous sections enlarge the original algebra of Abelian local gauge transformations found in [5] (expression (64) in [5]) to a non-Abelian case and unify into one multiplet particles with arbitrary spins and with linearly growing multiplicity. As we have seen, this leads to a natural generalization of the Yang-Mills

theory. The extended non-Abelian gauge transformations defined for the tensor gauge fields led to the construction of the appropriate field strength tensors and of the invariant Lagrangians. The proposed extension may lead to a natural inclusion of the standard theory of fundamental forces into a larger theory in which standard particles (vector gauge bosons, leptons and quarks) represent a low-spin subgroup of an enlarged family of particles with higher spins.

As an example of an extended gauge field theory with infinite many gauge fields, this theory *can be associated* with the field theory of the tensionless strings, because in our generalization of the non-Abelian Yang-Mills theory we essentially used the symmetry group which appears as symmetry of the ground state wave function of the tensionless strings [5, 4, 6]. Nevertheless I do not know how to derive it directly from tensionless strings, therefore one can not claim that they are indeed identical. The main reason is that the above construction, which is purely field-theoretical, has a great advantage of being well defined on and off the mass-shell, while the string-theoretical constructions have not been yet developed to the same level, because the corresponding vertex operators are well defined only on the mass-shell [6]. The tensor gauge field theory could probably be a genuine tensionless string field theory because of the common symmetry group, and it would be useful to understand, whether the string theory can fully reproduce this result. Discussion of the tensionless strings and related questions can also be found in [37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57].

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